Q1 (a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

(b) $f(x) = |x| \Rightarrow b_n = 0$, as odd part is zero.

$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx. \quad a_0 = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$

For $n>0$, $\frac{\pi a_n}{2} = \left[\frac{-x \sin nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n} dx$

$= \left[\frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}. \quad a_n = \begin{cases} \frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$

Thus $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos((2m-1)x)}{(2m-1)^2}$

(c) $\frac{1}{L}\int_{-L}^{L} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$..

(d) $\frac{1}{\pi}\int_{-\pi}^{\pi} |x|^2 dx = \frac{2\pi^2}{3} = \frac{1}{2}\frac{\pi^2}{4} + \frac{16}{\pi^2}\sum_{m=1}^{\infty}\left(\frac{1}{(2m-1)^2}\right)^2$

$\Rightarrow \quad \frac{\pi^4}{96} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \Rightarrow \pi^4 = 96\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}$

1402/ M14B. SOLUTIONS

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SUMMER 2005 SOLUTIONS

Q2 (a)

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

(b) $(\text{curl}\,(\underline{A}\times\underline{B}))_i = \varepsilon_{ijk}\,\partial_j\,(\underline{A}\times\underline{B})_k = \varepsilon_{ijk}\varepsilon_{klm}\,\partial_j(A_l B_m)$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,((\partial_j A_l)B_m + A_l(\partial_j B_m))$$

$$= (\partial_j A_i)B_j + A_i(\partial_j B_j) - (\partial_j A_j)B_i - A_j(\partial_j B_i)$$

$$= [\,\underline{B}\cdot\nabla\underline{A} + \underline{A}(\text{div}\,\underline{B}) - (\text{div}\,\underline{A})\underline{B} - \underline{A}\cdot\nabla\underline{B}\,]_i \quad \text{qed.}$$

(c) $\underline{A}\times\underline{B} = \underline{i}\times(x\underline{i} + y\underline{j} + z\underline{k}) = y\underline{k} - z\underline{j}$

$$\text{curl}\,(\underline{A}\times\underline{B}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & -z & y \end{vmatrix} = (1+1)\underline{i} + 0\underline{j} + 0\underline{k} = 2\underline{i}$$

$$\underline{B}\cdot\nabla\underline{A} - \underline{B}\,\nabla\cdot\underline{A} - \underline{A}\cdot\nabla\underline{B} + \underline{A}\,\nabla\cdot\underline{B}$$

$$= 0 + 0 - \frac{\partial}{\partial x}(x, y, z) + \underline{i}\;3$$

$$= -\underline{i} + 3\underline{i} = 2\underline{i} \quad \underline{\text{qed}}$$

Q 3 (a) Take $x = 3\cos\theta$, $y = 2\sin\theta$, $z = 0$

$\theta \in [0, 2\pi)$. $\quad d\underline{r} = (-3\sin\theta,\ 2\cos\theta,\ 0)\, d\theta$

$$\oint_C = \int_0^{2\pi} \left( (2\times 3\cos\theta - 2\sin\theta)(-3\sin\theta) + (2\times 2\sin\theta - 3\cos\theta)\, 2\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} (6\sin^2\theta + 6\cos^2\theta)\, d\theta \quad \text{as the other terms } \int = 0.$$

$$= \int_0^{2\pi} 6\, d\theta = 12\pi$$

(b) (i) $\displaystyle\int_C (y, z, yx) \cdot d\underline{r} = \int_C (t^3, t, t^5) \cdot (2t, 3t^2, 1)\, dt$

$$= \int_C (2t^4 + 3t^3 + t^5)\, dt = \frac{2}{5} + \frac{3}{4} + \frac{1}{6}$$

$$= \frac{79}{60}$$

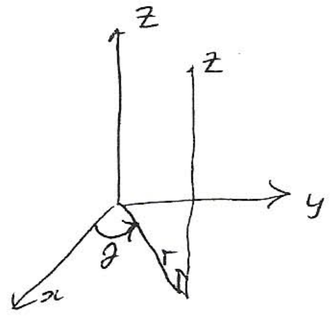
(ii) $\displaystyle\int_C = \int_C (t^2, t, t^3) \cdot (1, 2t, 1)\, dt$

$$= \int_C (t^2 + 2t^2 + t^3)\, dt = \frac{3}{3} + \frac{1}{4} = \frac{5}{4}$$

The end points in case (i) and (ii) are the same but the values of $\int_C$ are different, hence $\underline{G}$ is <u>not</u> conservative.

Q 4(a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$

(b)




$(x,y,z) = (r\cos\theta,\ r\sin\theta,\ z)$

$r \geq 0,\quad \theta \in [0,2\pi),\quad z \in (-\infty,\infty)$

(c) $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$= r.$

(d) $\displaystyle\int_V (1+z^3)\exp(x^2+y^2)\,dV = \int_V \exp(x^2+y^2)\,dV$ by symmetry

$\displaystyle = \int_0^1\int_0^{2\pi}\int_{-1}^1 r e^{r^2}\,dz\,d\theta\,dr = 4\pi\int_0^1 r e^{r^2}\,dr$

$\displaystyle = 2\pi\int_0^1 e^u\,du = 2\pi(e-1)$

Q5 (a)  $\int_V \operatorname{div} \underline{A}\, dv = \int_S \underline{A} \cdot \underline{n}\, ds$

where $\underline{n} =$ (unit outward normal to $S$) .

(b) $\underline{E} = -\nabla \varphi = -\nabla \frac{1}{r} = \frac{\underline{r}}{r^3}$

$\operatorname{div} \underline{E} = \nabla \cdot \left( \frac{\underline{r}}{r^3} \right) = \frac{\nabla \cdot \underline{r}}{r^3} - 3 \frac{\underline{r} \cdot \underline{r}}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0,$

if $|\underline{r}| \neq 0$.

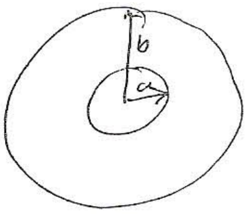
(c) $I = \int_{V_a} \operatorname{div} \underline{E}\, dV = \int_{S_a} \underline{E} \cdot \underline{n}\, ds$ , where $\underline{n} = \hat{\underline{r}}$

So $I = \int_{S_a} \left( \frac{\underline{r}}{r^3} \cdot \hat{\underline{r}} \right) ds = \int_{S_a} \left( \frac{1}{r^2} \right) ds$

$= \int_{S_a} \frac{1}{a^2}\, ds = \frac{1}{a^2} \int_{S_a} 1\, ds = \frac{1}{a^2} |S| = \frac{1}{a^2} 4\pi a^2 = 4\pi.$

$\underline{I = 4\pi}$

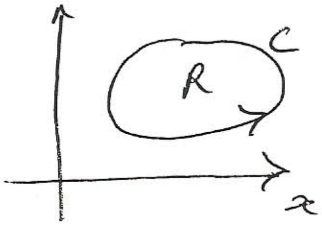
Consider the ball of radius $a$ inside the ball of radius $b > a$

 $\int_{V_a} + \int_{V_b - V_a} = \int_{V_b}$

But the region $a \leq |\underline{r}| \leq b$ has $\operatorname{div} \underline{E} = 0$,

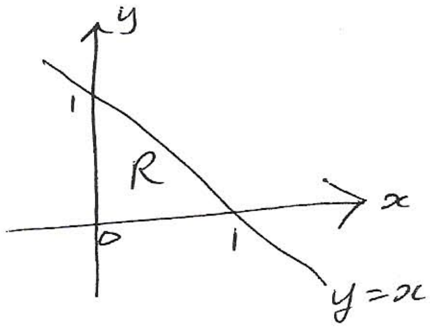
so $\int_{V_b - V_a} = 0$, giving $\int_{V_a} = \int_{V_b}$ q.e.d.

Q 6 (a)




$$\oint_C (P\,dx + Q\,dy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

(b)

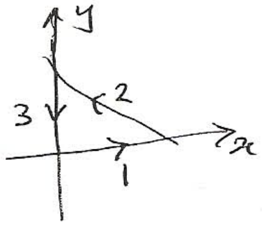



$$\int_R = \int_R (2(1+x+y) - x)\,dx\,dy$$

$$= \int_R (2 + x + 2y)\,dx\,dy = 2|R| + \int_R x + 2y\,dx\,dy$$

$$= 2 \times \tfrac{1}{2} + 3\int_R x\,dx\,dy \quad (\text{by symmetry}) = 1 + 3\int_0^1 \int_0^{1-y} x\,dx\,dy$$

$$= 1 + 3\int_0^1 \tfrac{1}{2}(1-y)^2\,dy = 1 + 3 \times \tfrac{1}{2} \times \tfrac{1}{3} = 1 + \tfrac{1}{2} = \tfrac{3}{2}.$$




On $C_1$, $y = 0$, so $\int_{C_1} (P\,dx + Q\,dy) = \int_{C_1} (0\,dx + Q\,0) = 0$

On $C_2$, $dx = -dy$ so $\int_{C_2} (P\,dx + Q\,dy) = \int_{C_2} (Q - P)\,dy$

$$= \int_0^1 [(1+1)^2 - y(1-y)]\,dy = 4 - \int_0^1 (y - y^2)\,dy = 4 - \tfrac{1}{2} + \tfrac{1}{3}$$

On $C_3$, $x = 0$, so $\int_{C_3} (P\,dx + Q\,dy) = \int_1^0 (P \times 0 + (1+y)^2)\,dy$

$$= \int_1^0 (1+y)^2\,dy = -\left[\frac{(1+y)^3}{3}\right]_0^1 = -\left(\frac{8}{3} - \frac{1}{3}\right) = -\frac{7}{3}$$

Thus $\int_C = 0 + 4 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{7}{3} = 4 - \tfrac{1}{2} - 2 = 3/2$

$= \int_R$ q.e.d.